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On the convergence of derivatives of Bernstein approximation

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Abstract

By differentiating a remainder formula of Stancu, we derive both an error bound and an asymptotic formula for the derivatives of Bernstein approximation.

Keywords: Bernstein approximation; Divided difference; Asymptotic formula; Error bound

1. Introduction

The Bernstein approximation $B_n(f)$ to a function $f : [0, 1] \to \mathbb{R}$ is the polynomial

$$B_n f(x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) p_{n,i}(x), \tag{1.1}$$

where $p_{n,i}$ is the polynomial of degree n,

$$p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad i = 0, \ldots, n.$$

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Bernstein [1] used this approximation to give the first constructive proof of the Weierstrass theorem. One of the many remarkable properties of Bernstein approximation is that derivatives of $B_n(f)$ of any order converge to corresponding derivatives of f; see Lorentz [7]. If $f \in C^k[0,1]$ for any $k \geqslant 0$, then

$$\lim_{n\to\infty} (B_n f)^{(k)} = f^{(k)} \qquad \text{uniformly on } [0,1].$$

Other remarkable properties are shape-preservation and variation-diminution [5].

These many properties can be viewed as compensation for the slow convergence of $B_n(f)$ to *f*. With $\|\cdot\|$ the max norm on $[0,1]$, the error bound

$$|B_n(f,x) - f(x)| \leqslant \frac{1}{2n} x(1-x)\|f''\|, \tag{1.2}$$

given in Chapter 10 of [4], shows that the rate of convergence is at least $1/n$ for $f \in C^2[0,1]$. On the other hand, the asymptotic formula

$$\lim_{n\to\infty} n\big(B_n f(x) - f(x)\big) = \tfrac{1}{2} x(1-x) f''(x), \tag{1.3}$$

due to Voronovskaya [9], shows that for $x \in (0,1)$ with $f''(x) \neq 0$, the rate of convergence is precisely $1/n$.

In this note we show that all derivatives of the operator B_n converge at essentially the same rate by extending both the error bound (1.2) and the Voronovskaya formula (1.3). Firstly, the error bound generalizes to:

Theorem 1. *If $f \in C^{k+2}[0,1]$ for some $k \geqslant 0$ then*

$$|(B_n f)^{(k)}(x) - f^{(k)}(x)| \leqslant \frac{1}{2n} \Big(k(k-1)\|f^{(k)}\| + k|1-2x|\|f^{(k+1)}\| + x(1-x)\|f^{(k+2)}\| \Big).$$

Secondly, Voronovskaya's formula can be 'differentiated':

Theorem 2. *If $f \in C^{k+2}[0,1]$ for some $k \geqslant 0$, then*

$$\lim_{n\to\infty} n\big((B_n f)^{(k)}(x) - f^{(k)}(x)\big) = \frac{1}{2} \frac{d^k}{dx^k} \left\{ x(1-x) f''(x) \right\},$$

uniformly for $x \in [0,1]$.

Thus the *k*th derivative of $B_n(f)$ converges at the rate of $1/n$ when the *k*th derivative of $x(1-x)f''(x)$ is non-zero.

We remark that after completion of this note, it was found that López–Moreno, Martínez–Moreno, and Muñoz–Delgado [6] very recently established Theorem 2 using a completely different approach.

2. Stancu's remainder formula

The traditional way to analyze the error $B_n(f) - f$ and indeed to derive both (1.2) and (1.3) is to substitute the Taylor expansion

$$f\left(\frac{i}{n}\right) = f(x) + \left(\frac{i}{n} - x\right) f'(x) + \cdots$$

into Eq. (1.1). To deal with derivatives of B_n we will instead borrow an idea from numerical differentiation [2]. As is well known, error formulas for numerical differentiation can be obtained from differentiating Newton's remainder formula for polynomial interpolation. This suggests finding an analogous remainder formula for Bernstein approximation and subsequently differentiating it. A natural remainder formula for this purpose is

$$B_n f(x) - f(x) = \frac{1}{n} x(1-x) \sum_{i=0}^{n-1} \left(\left[\frac{i}{n}, \frac{i+1}{n}, x \right] f \right) p_{n-1,i}(x). \tag{2.1}$$

Here $[x_0, x_1, \ldots, x_k] f$ denotes the kth order divided difference of f at the points $x_0, \ldots, x_k$, and we note that the right hand side of (2.1) is valid at least for f in $C^2[0, 1]$.

A more general form of this formula for the remainder in tensor-product bivariate Bernstein approximation was derived by Stancu [8], but does not appear to be too well known. It therefore seems worth offering the following proof, especially as it is shorter than the original in [8]. If one recalls the identity

$$\frac{1}{n} x(1-x) p'_{n,i}(x) = \left(\frac{i}{n} - x\right) p_{n,i}(x),$$

given in Chapter 10 of [4], Stancu's formula follows simply from

$$\begin{aligned}
B_n f(x) - f(x) &= \sum_{i=0}^{n} \left(f\left(\frac{i}{n}\right) - f(x) \right) p_{n,i}(x) \\
&= \sum_{i=0}^{n} \left[\frac{i}{n}, x \right] f \left(\frac{i}{n} - x \right) p_{n,i}(x) \\
&= \frac{1}{n} x(1-x) \sum_{i=0}^{n} \left(\left[\frac{i}{n}, x \right] f \right) p'_{n,i}(x) \\
&= x(1-x) \sum_{i=0}^{n-1} \left(\left[\frac{i+1}{n}, x \right] f - \left[\frac{i}{n}, x \right] f \right) p_{n-1,i}(x).
\end{aligned}$$

3. Error analysis

In what follows it will help to generalize the operator B_n to

$$B_{n,s,t} f(x) = \sum_{i=0}^{n-s} \left(\left[\frac{i}{n}, \ldots, \frac{i+s}{n}, \underbrace{x, \ldots, x}_{t} \right] f \right) p_{n-s,i}(x), \tag{3.1}$$

for any $s, t \geqslant 0$. We have $B_{n,0,0} = B_n$ and the remainder formula (2.1) can be written as

$$B_n f(x) - f(x) = \frac{1}{n} x(1-x) B_{n,1,1} f(x).$$

Differentiating this k times and using the Leibniz rule gives

$$\begin{aligned}(B_n f)^{(k)}(x) - f^{(k)}(x) = \frac{1}{n}\Big(&-k(k-1)(B_{n,1,1}f)^{(k-2)}(x) + k(1-2x)\\ &\times (B_{n,1,1}f)^{(k-1)}(x) + x(1-x)(B_{n,1,1}f)^{(k)}(x)\Big). \qquad (3.2)\end{aligned}$$

This leads us to study the derivatives of $B_{n,1,1} f$.

Lemma 1. *If $f \in C^{r+2}[0,1]$ for some $r \geqslant 0$ then*

$$(B_{n,1,1}f)^{(r)} = r! \sum_{j=1}^{r+1} j \frac{n-1}{n} \cdots \frac{n-j+1}{n} B_{n,j,r-j+2} f.$$

Proof. Using the formula (see Chapter 2 of [2])

$$\frac{d^r}{dx^r}\left[\frac{i}{n}, \frac{i+1}{n}, x\right] f = r! \left[\frac{i}{n}, \frac{i+1}{n}, \underbrace{x, \ldots, x}_{r+1}\right] f,$$

differentiation of (3.1) with $s = t = 1$ implies

$$\begin{aligned}(B_{n,1,1}f)^{(r)}(x) &= \sum_{i=0}^{n-1}\sum_{j=0}^{r}\binom{r}{j}(r-j)!\left(\left[\frac{i}{n}, \frac{i+1}{n}, \underbrace{x, \ldots, x}_{r-j+1}\right] f\right) p_{n-1,i}^{(j)}(x)\\ &= r! \sum_{j=0}^{r} \frac{(n-1)\ldots(n-j)}{j!} \sum_{i=0}^{n-j-1}\left(\Delta^j \left[\frac{i}{n}, \frac{i+1}{n}, \underbrace{x, \ldots, x}_{r-j+1}\right] f\right)\\ &\quad \times p_{n-j-1,i}(x), \qquad (3.3)\end{aligned}$$

where Δ is the forward difference operator w.r.t. i. Now notice that

$$\begin{aligned}\Delta\left[\frac{i}{n}, \frac{i+1}{n}, x, \ldots, x\right] f &= \left[\frac{i+1}{n}, \frac{i+2}{n}, x, \ldots, x\right] f - \left[\frac{i}{n}, \frac{i+1}{n}, x, \ldots, x\right] f\\ &= \frac{2}{n}\left[\frac{i}{n}, \frac{i+1}{n}, \frac{i+2}{n}, x, \ldots, x\right] f\end{aligned}$$

and continuing to apply Δ implies

$$\Delta^j \left[\frac{i}{n}, \frac{i+1}{n}, x, \ldots, x\right] f = \frac{2\,.\,3\ldots(j+1)}{n^j}\left[\frac{i}{n}, \ldots, \frac{i+j+1}{n}, x, \ldots, x\right] f.$$

Substituting this identity into Eq. (3.3) and replacing j by $j-1$ gives the result. $\square$

Due to Lemma 1, we have for $f \in C^{r+2}[0,1]$ and $r \geqslant 0$,

$$\|(B_{n,1,1}f)^{(r)}\| \leqslant r! \sum_{j=1}^{r+1} j \frac{\|f^{(r+2)}\|}{(r+2)!} = \frac{1}{2}\|f^{(r+2)}\|.$$

Theorem 1 now follows from applying this bound to Eq. (3.2). To prove Theorem 2 we study the convergence of the operators $B_{n,s,t}$.

Lemma 2. *If $f \in C^{s+t}[0,1]$ for some $s, t \geqslant 0$ then*

$$\lim_{n\to\infty} B_{n,s,t} f = \frac{f^{(s+t)}}{(s+t)!} \qquad \textit{uniformly on } [0,1].$$

Proof. We extend Davis's proof of Bernstein's theorem, namely the proof of Theorem 6.2.2 of [3]. Let $q := s+t$. Then for each i, $0 \leqslant i \leqslant n-s$, there is some ξ_i in the smallest interval containing $x, i/n, \ldots, (i+s)/n$ such that

$$\left[\frac{i}{n}, \ldots, \frac{i+s}{n}, \underbrace{x, \ldots, x}_{t}\right] f = \frac{f^{(q)}(\xi_i)}{q!}$$

and it is sufficient to show that

$$S_n := \sum_{i=0}^{n-s} (f^{(q)}(\xi_i) - f^{(q)}(x)) p_{n-s,i}(x) \to 0.$$

Let $\varepsilon > 0$. Since $f \in C^q[0,1]$, $\exists \delta > 0$ such that $|y-x| < \delta$ implies $|f^{(q)}(y) - f^{(q)}(x)| < \varepsilon$. Let I_n be the set of all i, $0 \leqslant i \leqslant n-s$, for which $x - \delta < i/n < (i+s)/n < x+\delta$, and split S_n into the two terms

$$C_n = \sum_{i \in I_n} (f^{(q)}(\xi_i) - f^{(q)}(x)) p_{n-s,i}(x),$$
$$D_n = \sum_{i \notin I_n} (f^{(q)}(\xi_i) - f^{(q)}(x)) p_{n-s,i}(x).$$

Now for $i \in I_n$ we clearly have $|\xi_i - x| < \delta$, and so

$$|C_n| \leqslant \sum_{i \in I_n} \varepsilon p_{n-s,i}(x) \leqslant \varepsilon.$$

Regarding D_n, notice that

$$\left|\frac{i}{n} - x\right| \leqslant \left|\frac{i}{n-s} - x\right| + \left|\frac{i}{n} - \frac{i}{n-s}\right| \leqslant \left|\frac{i}{n-s} - x\right| + \frac{s}{n}$$

and similarly

$$\left|\frac{i+s}{n} - x\right| \leqslant \left|\frac{i}{n-s} - x\right| + \left|\frac{i+s}{n} - \frac{i}{n-s}\right| \leqslant \left|\frac{i}{n-s} - x\right| + \frac{s}{n}$$

and therefore

$$\max\left\{\left|\frac{i}{n}-x\right|^2, \left|\frac{i+s}{n}-x\right|^2\right\} \leqslant \left(\frac{i}{n-s}-x\right)^2 + O(1/n),$$

uniformly for $x \in [0,1]$. It follows that

$$\begin{aligned}
|D_n| &\leqslant \frac{2}{\delta^2}\|f^{(q)}\| \sum_{i \notin I_n} \max\left\{\left|\frac{i}{n}-x\right|^2, \left|\frac{i+s}{n}-x\right|^2\right\} p_{n-s,i}(x) \\
&\leqslant \frac{2}{\delta^2}\|f^{(q)}\| \sum_{i=0}^{n-s} \left(\frac{i}{n-s}-x\right)^2 p_{n-s,i}(x) + O(1/n) \\
&= \frac{2}{(n-s)\delta^2}\|f^{(q)}\| x(1-x) + O(1/n).
\end{aligned}$$

Thus $\lim_{n\to\infty} |S_n| \leqslant \varepsilon$ for any $\varepsilon > 0$. $\square$

Due to Lemmas 1 and 2, we have for $f \in C^{r+2}[0,1]$ and $r \geqslant 0$,

$$\lim_{n\to\infty} (B_{n,1,1} f)^{(r)} = r! \sum_{j=1}^{r+1} j \frac{f^{(r+2)}}{(r+2)!} = \frac{f^{(r+2)}}{2} \qquad \text{uniformly on } [0,1],$$

and Theorem 2 follows from multiplying Eq. (3.2) by n and letting $n \to \infty$.

References

[1] S.N. Bernstein, Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités, Comm. Kharkov Math. Soc. 13 (1912) 1–2.

[2] S.D. Conte, C. de Boor, Elementary Numerical Analysis, McGraw-Hill, New York, 1980.

[3] P.J. Davis, Interpolation and Approximation, Dover, New York, 1975.

[4] R.A. DeVore, G.G. Lorentz, Constructive Approximation, Springer, Berlin, 1993.

[5] T.N.T. Goodman, Shape preserving representations, in: T. Lyche, L.L. Schumaker (Eds.), Mathematical Methods in Computer Aided Geometric Design, Academic Press, New York, 1989, pp. 333–357.

[6] A.J. López-Moreno, J. Martínez-Moreno, F.J. Muñoz-Delgado, Asymptotic expression of derivatives of Bernstein type operators, Rend. Circ. Mat. Palermo. Ser. II 68 (2002) 615–624.

[7] G.G. Lorentz, Zur theorie der polynome von S. Bernstein, Mate. Sbornik 2 (1937) 543–556.

[8] D.D. Stancu, The remainder of certain linear approximation formulas in two variables, SIAM J. Num. Anal. 1 (1964) 137–163.

[9] E. Voronovskaya, Détermination de la forme asymptotique d'approximation des fonctions par les polynômes de M. Bernstein, Dokl. Akad. Nauk SSSR (1932) 79–85.